Remarks on the Star-Triangle Relation in the Baxter-Bazhanov Model

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Abstract

In this letter we show that the restricted star-triangle relation introduced by Bazhanov and Baxter can be obtained either from the star-triangle relation of chiral Potts model or from the star-square relation which is proposed by Kashaev *et al* and give a response of the guess which is suggested by Bazhanov and Baxter in Ref. [3].

Keywords: Three-dimensional integrable lattice models, Baxter-Bazhanov model, Restricted star-triangle relations, Chiral Potts model, Star-square relation.

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1 Introduction

Recently much progress has been made in the three-dimensional integrable lattice models. Bazhanov and Baxter generalized the trigonometric Zamolodchikov model with two states [1] to the case of the arbitrary N states [2, 3]. The star-star relation and the star-square relation of this model are discussed in detail [3, 4, 5]. Boose and Mangazeev *et al* enlarge the integrable lattice model in three dimensions to the case where the weight functions are parameterized in terms of elliptic functions [6, 7, 8]. Just as the Yang-Baxter equations or the star-triangle relations play a central role in the theory of two-dimensional integrable models, the tetrahedron relations replace the Yang-Baxter equations as the commutativity conditions [9] for the three-dimensional lattice models. And the restricted star-triangle relations of the cubic lattice model introduced by Bazhanov and Baxter have the following form:

$$\sum_{l=0}^{N-1} \frac{w(v_2, a-l)}{w(v_1, -l)\gamma(b,l)} = \varphi_1(v_1, v_2) \frac{w(v_2', -b) w(v_2/(\omega v_1), a)}{w(v_1', a-b)}, \tag{1}$$

$$\sum_{l=0}^{N-1} \frac{w(v_3, -l)\gamma(b,l)}{w(v_4, a-l)} = \varphi_2(v_3, v_4) \frac{w(v_3', a-b)}{w(v_4', -b) w(v_4/v_3, a)}, \tag{2}$$

where φ_1 and φ_2 are scalar functions and

$$\frac{w(v,a)}{w(v,0)} = [\Delta(v)]^a \prod_{j=1}^{a} (1-\omega^j v)^{-1}, \quad v^N + \Delta^N = 1, \tag{3}$$

$$\omega = exp(2\pi i/N), \quad \omega^{1/2} = exp(i\pi/N), \quad \gamma(a,b) = \omega^{ab}. \tag{4}$$

v_i and $v_i'(i=1,2,3,4)$ satisfy

$$v_1' = \frac{v_2 \Delta(v_1)}{\omega v_1 \Delta(v_2)}, \quad \Delta(v_1') = \frac{\Delta(v_2/(\omega v_1))}{\Delta(v_2)}, \quad v_2' = \frac{\Delta(v_1)}{\Delta(v_2)}, \quad \Delta(v_2') = \frac{\omega v_1 \Delta(v_2/(\omega v_1))}{\Delta(v_2)}; \tag{5}$$

$$v_3' = \frac{v_4 \Delta(v_3)}{v_3 \Delta(v_4)}, \quad \Delta(v_3') = \frac{\Delta(v_4/v_3)}{\Delta(v_4)}, \quad v_4' = \frac{\Delta(v_3)}{\omega \Delta(v_4)}, \quad \Delta(v_4') = \frac{v_3 \Delta(v_4/v_3)}{\Delta(v_4)}. \tag{6}$$

Eq. (1) and Eq. (2) can be changed each other. Bazhanov and Baxter point out that it is quite possible that Eq. (1) is a particular case of a more general relation and γ is just a limiting value of a more complex function. The purpose of this letter is to give a response of it. In Sec. 2 the star-triangle relation of Baxter-Bazhanov model is obtained either from the star-triangle relation of the chiral Potts model or from the star-square relation introduced by Kashaev *et al*. In Sec. 3, the result is changed into the form of Eqs. (1) and (2). It should be note that the last relation in Eqs. (5) is different from the original one. The detail will be given also in Sec. 3.

2 The Star-Triangle Relation of Baxter-Bazhanov Model

As is well-known, the star-triangle relation of the chiral Potts model can be formulated as

$$\sum_{l=1}^{N} \bar{w}^{CP}_{qr}(m-l) w^{CP}_{pr}(n-l) \bar{w}^{CP}_{pq}(l-k) = R_{pqr} w^{CP}_{pq}(n-m) \bar{w}^{CP}_{pr}(m-k) w^{CP}_{qr}(n-k) \quad (7)$$

where

$$\frac{w^{CP}_{pq}(n)}{w^{CP}_{pq}(0)} = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}, \quad \frac{\bar{w}^{CP}_{pq}(n)}{\bar{w}^{CP}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \quad (8)$$

and

$$a_p^N + k' b_p^N = k d_p^N, \quad k' a_p^N + b_p^N = k c_p^N, \quad k^2 + k'^2 = 1. \quad (9)$$

Let

$$w(x, y, z|l) = \prod_{j=1}^{l} \frac{y}{z - x\omega^j}, \quad x^N + y^N = z^N, \quad (10)$$

$$w_{pq}(n) \equiv w(\omega^{-1} c_p b_q, d_p a_q, b_p c_q | n) \quad (11)$$

and define the map R as

$$R: \quad (a_p, b_p, c_p, d_p) \longrightarrow (b_p, \omega a_p, d_p, c_p). \quad (12)$$

When we set $a_p = d_r = 0$, the following relations are obtained:

$$\frac{w^{CP}_{pr}(n)}{w^{CP}_{pr}(0)} = w_{pR(r)}(n), \qquad \frac{\bar{w}^{CP}_{pr}(n)}{\bar{w}^{CP}_{pr}(0)} = \frac{1}{w_{pr}(-n)}; \quad (13)$$

$$\frac{w^{CP}_{pq}(n)}{w^{CP}_{pq}(0)} = w_{pR(q)}(n), \qquad \frac{\bar{w}^{CP}_{pq}(n)}{\bar{w}^{CP}_{pq}(0)} = \frac{1}{w_{pq}(-n)}; \quad (14)$$

$$\frac{w^{CP}_{qr}(n)}{w^{CP}_{qr}(0)} = w_{R^{-1}(q)r}(-n), \quad \frac{\bar{w}^{CP}_{qr}(n)}{\bar{w}^{CP}_{qr}(0)} = \frac{1}{w_{qr}(-n)}. \quad (15)$$

By taking account of the star-triangle equation (7) of chiral Potts model we get

$$\sum_{l=1}^{N} \frac{w_{pR(r)}(n-l)}{w_{qr}(l-m) w_{pq}(k-l)} = R'_{pqr} \frac{w_{pR(q)}(n-m) w_{R^{-1}(q)r}(k-n)}{w_{pr}(k-m)} \quad (16)$$

with $a_p = d_r = 0$ where R'_{pqr} is a scalar function. This is just the star-triangle equation of Baxter-Bazhanov model. If we set $a_p = c_r = 0$, similarly we have

$$\sum_{l=1}^{N} \frac{w_{pR(r)}(n+l)}{w_{R(q)R(r)}(m+l) w_{pq}(k+l)} = \bar{R}'_{pqr} \frac{w_{pR(q)}(n-m) w_{qR(r)}(n-k)}{w_{R(p)R(r)}(m-k)} \quad (17)$$

where $\bar{R}'_{pqr}$ is also a scalar function. Both of the above two equations can be changed into the form of Eqs. (1) and (2). It will be discussed in Sec. 3. Now we give the connection between Eq. (16) and the star-square relation in Baxter-Bazhanov model. Let

$$w(x,y,z|l) = (y/z)^l w(x/z|l), \quad \Phi(a-b) = \omega^{(a-b)(N+a-b)/2}. \tag{18}$$

As the version of Kashaev *et al*, the star-square relation can be written as [4]

$$\begin{aligned}
&\left\{ \sum_{\sigma \in Z_N} \frac{w(x_1,y_1,z_1|a+\sigma)w(x_2,y_2,z_2|b+\sigma)}{w(x_3,y_3,z_3|c+\sigma)w(x_4,y_4,z_4|d+\sigma)} \right\}_0 \\
&\quad = \frac{(x_2y_1/x_1z_2)^a(x_1y_2/x_2z_1)^b(z_3/y_3)^c(z_4/y_4)^d}{\Phi(a-b)\omega^{(a+b)/2}} \\
&\qquad \times \frac{w(\omega x_3x_4z_1z_2/x_1x_2z_3z_4|c+d-a-b)}{w\left(\frac{x_4z_1}{x_1z_4}|d-a\right)w\left(\frac{x_3z_2}{x_2z_3}|c-b\right)w\left(\frac{x_3z_1}{x_1z_3}|c-a\right)w\left(\frac{x_4z_2}{x_2z_4}|d-b\right)},
\end{aligned} \tag{19}$$

where the subscript "0" after the curly brackets indicates that the l. h. s. of the above equation is normalized to unity at zero exterior spins and the constraint condition $y_1y_2z_3z_4/(z_1z_2y_3y_4) = \omega$ should be imposed owing to spin $\sigma \in Z_N$ but r. h. s. of the above equation is independent of σ. Set

$$\begin{array}{lll}
x_1 = c_q b_r, & y_1 = \omega d_q a_r, & z_1 = b_q c_r, \\
x_2 = c_p a_q, & y_2 = d_p b_q, & z_2 = b_p d_q, \\
x_3 = \omega_{-1} c_p b_r, & y_3 = d_p a_r, & z_3 = b_p c_r, \\
x_4 = 0, & y_4 = z_4. &
\end{array} \tag{20}$$

By considering the "inversion" relation [4, 5]

$$\sum_{k \in Z_N} \frac{w(x,y,z|k,l)}{w(x,y,\omega z|k,m)} = N\delta_{l,m} \frac{1-z/x}{1-z^N/x^N} \tag{21}$$

where $\delta_{l,m}$ is the Kronecker symbol on Z_N we get the equation (16) from the star-square relation (19). Eq. (17) can be obtained similarly.

3 Discussion

Firstly, Eq. (16) can be changed into the form of Eq. (1) and Eq. (2) by using the notations

$$v_1 = \frac{c_p b_q}{\omega b_p c_q}, \quad v_2 = \frac{b_q c_r}{c_q b_r}, \quad \Delta(v_1) = \frac{d_p a_q}{b_p c_q}, \quad \Delta(v_2) = \frac{\omega^{1/2} d_q a_r}{c_q b_r},$$

$$v'_1 = \frac{a_q c_r}{d_q b_r}, \quad v'_2 = \frac{c_p a_q}{b_p d_q}, \quad \Delta(v'_1) = \frac{\omega^{1/2} c_q a_r}{d_q b_r}, \quad \Delta(v'_2) = \frac{d_p b_q}{b_p d_q}, \tag{22}$$

$$\Delta(\frac{v_2}{\omega v_1}) = \frac{\omega^{1/2} d_p a_r}{c_p b_r},$$

and

$$\begin{array}{llll} v_3 = \frac{b_q c_r}{c_q b_r}, & v_4 = \frac{c_p b_q}{\omega b_p c_q}, & \Delta(v_3) = \frac{\omega^{1/2} d_q a_r}{c_q b_r}, & \Delta(v_4) = \frac{d_p a_q}{b_p c_q}, \\ v'_3 = \frac{d_q b_r}{\omega a_q c_r}, & v'_4 = \frac{b_p d_q}{\omega c_p a_q}, & \Delta(v'_3) = \frac{c_q a_r}{a_q c_r}, & \Delta(v'_4) = \frac{d_p b_q}{\omega^{1/2} c_p a_q}; \end{array} \tag{23}$$

respectively, with $d_p b_r = \omega^{1/2} c_p a_r$. v_i and v'_i $(i = 1, 2, 3, 4)$ satisfy Eqs. (5) and (6). Here we show that the last relation in Eq. (5) is correct and this relation is different from the one in Ref. [3]. Eq. (17) can be also changed into Eq. (1) by setting

$$\begin{array}{llll} v_1 = \frac{b_p d_q}{\omega c_p a_q}, & v_2 = \frac{d_q a_r}{a_q d_r}, & \Delta(v_1) = \frac{d_p b_q}{\omega^{1/2} c_p a_q}, & \Delta(v_2) = \frac{c_q b_r}{a_q d_r}, \\ v'_1 = \frac{c_p b_q}{\omega b_p c_q}, & v'_2 = \frac{b_q d_r}{\omega c_q a_r}, & \Delta(v'_1) = \frac{d_p a_q}{b_p c_q}, & \Delta(v'_2) = \ frac d_q b_r \omega^{1/2} c_q a_r, \end{array} \tag{24}$$

$$\Delta(\frac{v_2}{\omega v_1}) = \frac{d_p b_r}{b_p d_r},$$

with $c_p b_r = \omega^{1/2} d_p a_r$. Similarly Eq. (2) can be obtained easily from Eq. (17). In fact, each of the relations (16), (17) is a corollary of another by taking account of the "inversion" relation (21).

As a conclusion, in this letter we get the star-triangle relation of the Baxter-Bazhanov model from the star-triangle relation of the chiral Potts model and give a response to the guess proposed by Bazhanov and Baxter. And the connection is found between the star-triangle relation and the star-square relation in the Baxter-Bazhanov model.

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